Tonga National Form Seven Certificate
MATHEMATICS with CALCULUS
2016

QUESTION and ANSWER BOOKLET
Time allowed: Three Hours

INSTRUCTIONS
1. Write your Student Enrolment Number (SEN) on the top right hand corner of this booklet.
2. ANSWER ALL QUESTIONS. Write your answers for each question in the spaces provided in this booklet.
3. If you need more space for answers, ask the Supervisor for extra paper. Write your SEN on all extra sheets used and clearly number the questions. Attach the extra sheets at the appropriate places in this booklet.
4. Show all working. Unless otherwise stated, numerical answers correct to three significant figures will be adequate.

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Check that this booklet contains pages 2-23 in the correct order and that none of these pages is blank.

A 4-page booklet (No. 108/2) containing mathematical formulae and tables has been provided.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION
SECTION A ALGEBRA

Question 1

Given \( z = \sqrt{6} - \sqrt{2}i \) and \( w = 2\sqrt{6} + 2\sqrt{2}i \).

a. Evaluate the following, expressing your answers in the form \( a + ib \).

i. \( z + w \)

ii. \( \frac{z}{w} \)

b. Express \( z \) in the form \( rcis\theta \) where \( -\pi < \theta \leq \pi \).
c. Evaluate the following, expressing your answers in polar form.

i. \( z^3 \)

ii. \( \sqrt[3]{z} \)

**QUESTION 2**

Given that \( \log_a 2 = x \) and \( \log_a 3 = y \). Write down an expression for \( \log_a 24 \) in terms of \( x \) and \( y \).
QUESTION 3

The remainder when a polynomial \( p(x) \) is divided by \( x^2 - 4 \) is \( 2x - 1 \). What is the remainder when \( p(x) \) is divided by \( (x + 2) \)?

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QUESTION 4

Find real numbers \( P, Q \) and \( R \) such that

\[
\frac{1}{x(x^2 + 2)} = \frac{P}{x} + \frac{Qx + R}{x^2 + 2}
\]

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QUESTION 5

A drug is used to control a medical condition. It is known that the quantity, $Q$ in milligrams, of the drug remaining in the body after $t$ hours satisfies an equation of the form

$$Q = 6e^{-kt},$$

where $k$ is a constant.

a. Find the value of $k$ to four decimal places, if half of the initial dose remains in the body after 15 hours.

b. When will one-eighth of the initial dose remain?
QUESTION 6
Find the coefficient of $x^9 y^{10}$ in the expansion of $(2x-3y^2)^{14}$

QUESTION 7
Simplify $\frac{8^\frac{3}{2}(6a)^{-\frac{3}{2}}}{\sqrt{3a}}$
QUESTION 8

Use principle of mathematical induction to prove

\[
\sum_{k=0}^{n} (a + kb) = \frac{1}{2} (n+1)(2a + nb)
\]

where \(a\) and \(b\) are constants.
SECTION B  
TRIGONOMETRY

QUESTION 1

Find the exact value of;

i. \( \cot\left(\frac{2\pi}{3}\right) \)

ii. \( \cos(75^\circ) \)

QUESTION 2

The angle \( \theta \) satisfies \( \sin \theta = \frac{5}{13} \) and \( \frac{\pi}{2} < \theta < \pi \). Use the double angle formulae to find the value of \( \sin 2\theta \)
QUESTION 3

Write \( \frac{\cos(3x) - \cos x}{\cos x + \cos(3x)} \) as a product of two tangent functions.

QUESTION 4

Prove the trigonometric identity \( \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \).
QUESTION 5

The tide or depth of the ocean near the shore changes throughout the day. The depth of the Bay of Fundy, site of highest tides in the world, can be modeled by

\[ d = 35 - 28 \cos \frac{\pi t}{6.2} \]

where \( d \) is the water depth in feet and \( t \) is the time in hours. If \( t = 0 \) represent 12:00 A.M. on a particular day, what time is the first high tide occur on this particular day?
QUESTION 6

Solve the equation: \(1 - \cos x = \sqrt{3} \sin x\) in the interval \(0 \leq x < 2\pi\).

**Hint:** Square both sides and use appropriate identities.
SECTION C

Differentiation

QUESTION 1
Write down the value(s) of \( x \) for which the function \( f(x) = \frac{x + 1}{x^2 - 1} \) is discontinuous.

QUESTION 2
Differentiate the following functions with respect to \( x \). You do not need to simplify the answers.

a. \( \sqrt{x} - \frac{1}{x} + e^x \)

b. \( e^x - \ln x \)

c. \( \cos^3(2x - \pi) \)
d. \( \frac{\ln 2x}{x} \)

**QUESTION 3**

Differentiate \( p(x) = x - 2x^2 \) using the definition \( p'(x) = \lim_{h \to 0} \frac{p(x + h) - p(x)}{h} \)

**Skill level 2**

| 3 | 2 | 1 | 0 | NR |

**Skill level 3**

| 3 | 2 | 1 | 0 | NR |
QUESTION 4

Evaluate \( \lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2} \)

QUESTION 5

A particle \( P \) is moving along the \( x \) axis. Its position at time \( t \) seconds is given by

\[ x = 2 \sin t - t, \quad t \geq 0. \]

a. Find an expression for the velocity of the particle.

b. Write down an expression for the acceleration of the particle.
**QUESTION 6**

Find the equation of the tangent to the circle \((x-6)^2 + (y-1)^2 = 25\) at the point \((3,5)\).

**QUESTION 7**

A stone dropped into a still water sends out circular ripple whose radius increases at a constant rate of \(2m/s\). How fast is the area, enclosed by the ripple, increasing when the radius of the ripple is \(10m\)?
A cylinder of radius $x$ and height $2h$ is to be inscribed in a sphere of radius $R$ centered at $O$. The volume of the cylinder is

$$V = 2\pi h (R^2 - h^2).$$

Show that the volume of the cylinder is at maximum when $h = \frac{R}{\sqrt{3}}$. 

**Skill level 4**

| 4 | 3 | 2 | 1 | 0 | NR |
QUESTION 9

Sketch the curve \( y = x^3 - 3x^2 \). Label the intercepts, stationary points and the point(s) of inflexion.
SECTION D  
Integration

QUESTION 1
Evaluate \[ \int_{0}^{\ln 2} 2e^{2x} \, dx \]

QUESTION 2
Perform the following indefinite integrals. You do not need to simply the answers.

a. \[ \int \left( \sqrt{x^2} - \frac{1}{x^2} + \pi \right) \, dx \]

b. \[ \int x(x^2 - 1)^{10} \, dx \]
c. \( \int \sin(7x) \cos(3x) \, dx \)

d. \( \int \frac{2x + 1}{x - 1} \, dx \)
QUESTION 3

Use the trapezoidal rule with 3 function values to find an approximation to \( \int_{1}^{3} \ln x \, dx \).

QUESTION 4

Given that: \( \sec^2 x + \sec x \tan x = \frac{1}{1 - \sin x} \). Evaluate \( \int_{0}^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} \).
QUESTION 5

A cup of coffee with an initial temperature of 80°C is placed in a room with a constant temperature of 22°C. After 10 minutes the temperature of the cup of coffee drops to 60°C.

The rate at which the temperature, $T$, of the cup of coffee cools down satisfies the differential equation

$$\frac{dT}{dt} = k(T - 22)$$

How long does it take for the temperature of the cup of coffee to drop to 40°C? Express your answer to the nearest minute.
QUESTION 6

The diagram shows the graphs of the functions \( f(x) = 4x^3 - 4x^2 + 3x \) and \( g(x) = 2x \) intersecting at the origin \( O \) and at \( T\left(\frac{1}{2}, 1\right) \) Find the area of the region bounded by the graphs of \( f(x) \) and \( g(x) \).
QUESTION 7

The region bounded by the $x$-axis, the $y$-axis and the parabola $y = (x - 2)^2$ is rotated about the $y$-axis to form a solid. Calculate the volume of the solid.